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FEATURES OF COMBINED INSTABILITY OF A CHARGED INTERFACE BETWEEN MOVING MEDIA

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In solving the problem on the propagation of small-amplitude waves on a charged horizontal interface between two liquids, in the dispersion determinant derived, a term that is nonlinear with respect to the wave frequency combination and the velocity ratio between the upper and lower liquids has been obtained. The structure and the physical meaning of the equation obtained have been discussed. The notion of effective surface tension has been introduced.

Introduction. Numerous investigations in the field of surface electrohydrodynamics intensively developed at the end of the 20th century have led to the conclusion that the principal results in the linear approximation have already been obtained and primary consideration should be given to the nonlinear statement of the problem. The theses defended in recent years (e.g., [1]) confirm this tendency. However, the bifurcation branchings discovered in due course [2, 3] have shown that even in the linear approximation the problem is many-sided and presupposes the presence of a large number of effects. Continuation of investigations on these lines is attractive due to the fact that the results obtained in this case have a more visual, graphic, and easy-to-interpret form than those obtained in the nonlinear theory [4, 5].

It is known that the horizontal interface between two fluid media is subjected to instabilities of different kinds. For instance, in the gravity field the interface of a heavy liquid over a light one is subjected to Rayleigh–Taylor (R–T) instability. One encounters it in modern facilities for inertial thermonuclear fusion. When one of the liquids or a gas has a background horizontal motion, then the Kelvin–Helmholtz (K–H) instability, by which the excitation by wind of waves on a smooth water surface can be explained, arises. In the case where an external electric field is applied perpendicularly to the interface, the question is of the so-called Frenkel–Tonks (F–T) instability, which is always in the field of vision of researchers because of its wide occurrence, e.g., "Saint Elmo's fire," radiation of clouds before a storm, electrohydrodynamic ion emitters, etc.

The description of all kinds of instabilities, which have already become classical, is widely represented in the literature (see, e.g., [7, 8]). Recently there have been attempts to bring all these instabilities together into a general model, and often not only Newtonian liquids have been considered. However, as a rule, all results were reduced to particular or limiting cases, and at best, a numerical calculation was made, i.e., it has been impossible to bring the F–T, R–T, and K–H instabilities together into the framework of the general analytical theory.

Derivation of the Dispersion Equation. Let us derive the dispersion equation (DE) for small-amplitude waves on a charged horizontal interface between two immiscible liquids: the upper liquid is assumed to be ideal and the lower one, viscous; on the interface the tangential velocity discontinuity remains.

For the upper liquid, we introduce the velocity potential Φ' for waves propagating in the positive direction of the $0x$-axis in the form

$$\Phi' = F \exp[-kz] \exp[i(kx - \omega t)] + Ux \,. \qquad (1)$$

On the interface we have

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$$v_z = \frac{\partial \xi}{\partial t}. \tag{2}$$

Note that from the equality $v_n = v_n'$ under the conditions of our problem it does not follow that $v_z = v_z'$, since

$$v_z' = U\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial t}. \tag{3}$$

For the lower liquid (of arbitrary viscosity), we seek the solution of the equations of motion (Navier–Stokes equation for waves with a small amplitude)

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2}\right), \quad \frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \upsilon\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2}\right) - g, \quad \frac{\partial v_x}{\partial z} + \frac{\partial v_x}{\partial z} = 0,$$

depending on t and x as $\exp[i(kx - \omega t)]$ and attenuating into the depth of the liquid ($z > 0$) in the form

$$v_x = \exp[i(kx - \omega t)]\,(A\exp[kz] + B\exp[lz]), \quad v_z = \exp[i(kx - \omega t)]\,(C\exp[kz] + D\exp[lz]),$$

$$p = \frac{\rho\omega}{k}A\exp[kz]\exp[i(kx - \omega t)] - \rho g z, \quad C = -iA, \quad D = -i\frac{k}{l}B, \quad l^2 = k^2 - \frac{i\omega}{\upsilon}, \tag{4}$$

where A, B, C, D, F are some constants.

The boundary conditions on the interface are as follows:

1. Shear stresses are equal to zero: $\sigma_{xz} = 0$, i.e.,

$$\eta\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) = 0. \tag{5}$$

2. For pressures on the interface, the equality

$$-p + 2\eta\frac{\partial v_z}{\partial z} - \gamma\frac{\partial^2 \xi}{\partial x^2} - 4\pi\sigma^2 k\xi = \rho'\frac{\partial \Phi'}{\partial t} + \rho' g\xi + \frac{\rho'}{2}(v'^2 - U^2) \tag{6}$$

holds.

In the case of $U = 0$, the procedure of reducing the initial system (1)–(6) to a dispersion equation is analogous to that described in [7]. Substituting expressions (1) and (2) into (3) in view of the fact that $v_z' = \partial\Phi'/\partial z$, for a solution of the form of (4), we obtain the first equation for constants C, D, F:

$$C + D - \frac{k\omega}{Uk - \omega}F = 0. \tag{7}$$

From Eq. (5) for the solution of the form of (4) we have

$$2C - (2 - \chi)D = 0, \tag{8}$$

where $\chi = \dfrac{i\omega}{\upsilon k^2}$. Differentiate equality (6) with respect to time and substitute the expressions for p, v_z, taking into account $v'^2 = v_x'^2 + v_z'^2 = \left(\dfrac{\partial \Phi'}{\partial x}\right)^2 + \left(\dfrac{\partial \Phi'}{\partial z}\right)^2$. Then

$$C\left(\frac{\chi}{2} - 1 + W(k)\right) + D\left(W(k) - \frac{l}{k}\right) + \frac{i\rho'}{\rho}\frac{(Uk - \omega)}{2\upsilon k} F = 0\,, \tag{9}$$

$$W(k) = \frac{\gamma k^2 - 4\pi\sigma^2 k + (\rho - \rho')\,g}{2\eta i\omega k}\,.$$

Bring Eqs. (7)–(9) into one system. The requirement for nontriviality of the solution for *C*, *D*, *F* is given by a DE in the form of the 3d-order dispersion determinant:

$$\begin{vmatrix} 1 & 1 & -\dfrac{k\omega}{Uk - \omega} \\ \left\{\dfrac{\chi}{2} - 1 + W(k)\right\} & \left\{W(k) - \dfrac{l}{k}\right\} & \dfrac{i\rho'}{\rho}\dfrac{(Uk - \omega)}{2\upsilon k^2} \\ 2 & 2 - \chi & 0 \end{vmatrix} = 0\,. \tag{10}$$

Equation (10) is just the sought DE written in implicit form; the third line has already been divided by $\eta \neq 0$. The columns of the dispersion determinant correspond to the coefficients before *C*, *D*, *F* in Eqs. (7)–(9); the second column of the dispersion determinant corresponds to the coefficients before *D*, i.e., it is determined by the presence of viscosity; the third column is connected with the moving upper ideal liquid. The first line presents the coefficients before constants *C*, *D*, *F* in the kinematic relation on the interface between the liquids; the second line shows the analogous coefficients for the dynamic condition — equality of pressures in the liquid on the interface; the second line gives the coefficients in the second dynamic condition — the absence of shear stresses on the ideal–nonideal liquid interface.

Let us develop the dispersion determinant (10). Upon elementary manipulations we obtain

$$(2 - \chi)^2 + \Omega(k) + K(k, \omega) = 4\sqrt{1 - \chi}\,, \tag{11}$$

$$K(k, \omega) = -\alpha\frac{(Uk - \omega)^2}{\upsilon^2 k^4}\,, \quad \Omega(k) = \frac{\gamma k^2 - 4\pi\sigma^2 k + (\rho - \rho')\,g}{\rho\upsilon^2 k^3}\,, \quad \alpha = \frac{\rho'}{\rho}\,.$$

Note that the DE (11) is analogous to that obtained earlier in [4] for the case of the absence of the upper liquid or the absence of its motion with the only difference — the addition of $K(k, \omega)$ nonlinear with respect to the combination of ω and Uk, i.e., already at small values of α and U qualitative differences in the physical picture of the investigated process should be observed. From the DE (11) all kinds of hydrodynamic instabilities of the liquid surface under consideration follow: F–T: $\alpha = 0$, $\upsilon = 0$; K–H: $\sigma = 0$, $\upsilon = 0$; R–T: $\sigma = 0$, $\upsilon = 0$, $\alpha > 1$ [6, 7].

Let us introduce $\omega = \omega' + i\beta$, where β is the attenuation coefficient and ω' is the cyclic frequency of the wave. Having separated in Eq. (11) the real and imaginary parts, we get

$$\begin{gathered} ((2 - \tilde\beta)^2 + \Omega(k) - \tilde\omega^2 + \alpha(\tilde\beta^2 - (\theta - \tilde\omega)^2))\,(\tilde\omega(2 - \beta) + \alpha\tilde\beta(\theta - \tilde\omega)) = 4\tilde\omega\,, \\ ((2 - \tilde\beta)^2 + \Omega(k) - \tilde\omega^2 + \alpha(\tilde\beta^2 - (\theta - \tilde\omega)^2))^2 - 4(\tilde\omega(2 - \beta) + \alpha\tilde\beta(\theta - \tilde\omega))^2 = 16(1 - \tilde\beta)\,, \end{gathered} \tag{12}$$

where $\tilde\omega = \dfrac{\omega'}{\upsilon k^2}$; $\tilde\beta = \dfrac{\beta}{\upsilon k^2}$; $\theta = \dfrac{U}{\upsilon k}$.

Figure 1 shows the dependences of ω and β on *k* for various values of α ($\alpha = 0$ for Fig. 1a, b and $\alpha = 0.15$ for Fig. 1c, d) at the following input data: $\gamma = 0.62$ J/m^2, $\upsilon = 5$ m^2/sec, $\rho = 10^3$ kg/m^3, $U = 1$ m/sec, $\sigma = 60$ μC/m^2.

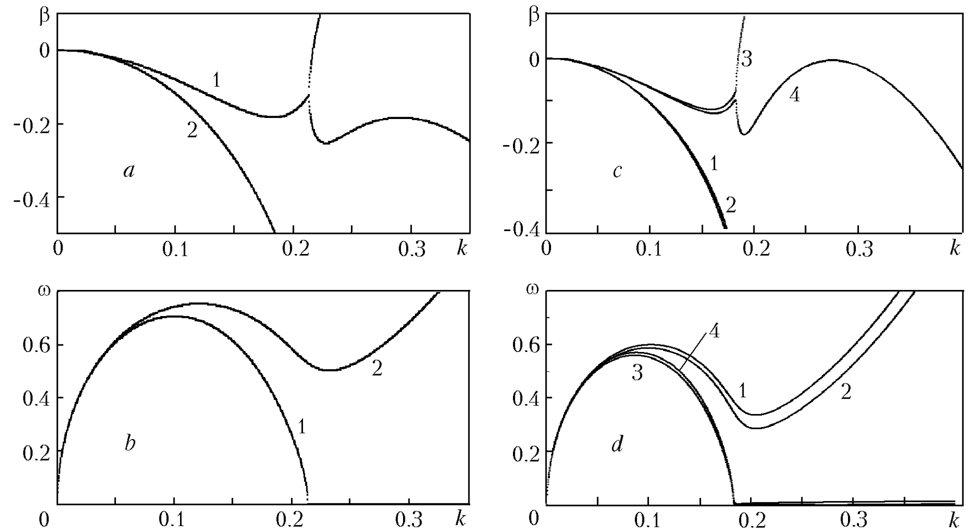

Fig. 1. Attenuation coefficient (a, c) and cyclic frequency (b, d) versus the wave vector for the case without the upper liquid (a, b) and with a moving upper liquid (c, d). Numbers denote the corresponding frequency curves and attenuation coefficients for similar modes. ω, sec^{-1}; k, m^{-1}; β, sec^{-1}.

Investigation of the system obtained in explicit form is difficult and presupposes numerical calculations, as has already been done earlier ([9] and the references therein). However, some conclusions can already be drawn from these equations as well. In particular, at $\alpha \neq 0$, $\theta \neq 0$, no matter how small, branching for waves with $\tilde{\omega} = 0$ (aperiodic wave) are observed. Indeed, at $\tilde{\omega} = 0$ (12) goes over into the system

$$\alpha\tilde{\beta}\theta\,((2-\tilde{\beta})^2 + \Omega\,(k) + \alpha\,(\tilde{\beta}^2 - \theta^2)) = 0\,,$$

$$((2-\tilde{\beta})^2 + \Omega\,(k) + \alpha\,(\tilde{\beta}^2 - \theta^2))^2 - 4\alpha^2\tilde{\beta}^2\theta^2 = 16\,(1-\tilde{\beta})\,,$$

for which we find two solutions:

$$\tilde{\beta}_1 = 2\,\frac{1-\sqrt{1-\varepsilon}}{\varepsilon}\,,\quad \tilde{\beta}_2 = 2\,\frac{1+\sqrt{1-\varepsilon}}{\varepsilon}\,,\quad \varepsilon = (\alpha\theta)^2 = \left(\frac{\rho'}{\rho}\,\frac{U}{\upsilon k}\right)^2,$$

$$\Omega_{1,2}\,(k) = \alpha\,(\theta^2 - \tilde{\beta}_{1,2}^2) - (2-\tilde{\beta}_{1,2})^2\,.$$

It is seen that the first solution is regular, but besides the regular solution there also appear a singular solution $\tilde{\beta}_2$.

Effective Surface Tension Coefficients. For the case there the lower liquid is ideal, the dispersion equation (11) takes on the form

$$\omega^2 - \frac{2\rho' U k}{\rho + \rho'}\,\omega + k\left(\frac{\rho' - \rho}{\rho' + \rho}\,g + \frac{(\rho' U + 4\pi\sigma^2)}{\rho' + \rho}\,k - \frac{\gamma k^2}{\rho' + \rho}\right) = 0\,.$$

For the wave to be stable, the condition $\mathrm{Im}(\omega) = 0$ should be fulfilled for all values of k, i.e., satisfaction of the inequality

$$\left(\frac{\rho\rho'}{\rho + \rho'}\,U^2 + 4\pi\sigma^2\right)^2 - 4\gamma g\,(\rho - \rho') < 0$$

is necessary and sufficient.

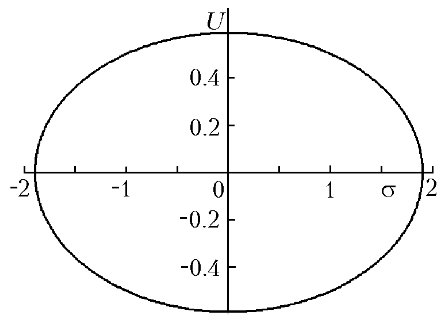

Fig. 2. Instability ellipse.

At the moment of going to the unstable state we have the equation

$$\left(\frac{\rho\rho'}{\rho+\rho'}U^2+4\pi\sigma^2\right)=2\sqrt{\gamma g\,(\rho-\rho')}\,. \tag{13}$$

At $U = 0$ the change-over to the F–T instability and at $\sigma = 0$ to the K–H instability is observed.

The geometrical interpretation represents the instability ellipse

$$\frac{U^2}{U_0^2}+\frac{\sigma^2}{\sigma_0^2}=1\,, \tag{14}$$

where

$$U_0=\sqrt{\frac{2\,(\rho+\rho')\sqrt{\gamma g\,(\rho-\rho')}}{\rho\rho'}}\;;\quad \sigma_0=\frac{\sqrt[4]{\gamma g\,(\rho-\rho')}}{\sqrt{2\pi}}\,.$$

The instability ellipse for a liquid with parameters $\alpha = 0.15$, $\gamma = 0.062$ J/m^2, $\rho = 10^3$ kg/m^3 is given in Fig. 2. The points inside the instability ellipse correspond to stable states, and those in the outer region to unstable states; the extreme upper and lower points correspond to the classical K–H instability; the extreme right and left points correspond to the classical F–T instability.

Let us consider the instability ellipse (14) and introduce the effective surface tension coefficient according to Frenkel–Tonks:

$$16\pi^2\sigma^4=4g\,(\rho-\rho')\,\gamma_e^{\sigma} \tag{15}$$

and according to Kelvin–Helmholtz:

$$\frac{\rho^2\rho'^2}{(\rho+\rho')^2}U^4=4g\,(\rho-\rho')\,\gamma_e^{U}\,. \tag{16}$$

From Eq. (13) and expressions (15), (16) for γ_e^{σ} and γ_e^{U} it is seen that they are related to γ by the relations

$$\gamma_e^{\sigma}=\gamma\,(1+\delta)^{-2}\,,$$

$$\gamma_e^{U}=\gamma\left(1+\frac{1}{\delta}\right)^{-2}, \tag{17}$$

where $\delta = \dfrac{\rho\rho' U^2}{4\pi\sigma^2(\rho + \rho')}$. At $\delta << 1$ we can expand (17) into powers δ, which yields

$$\gamma_e^{\sigma} \approx \gamma\,(1 - 2\delta) = \gamma - \frac{\rho\rho'\gamma}{2\pi\sigma^2\,(\rho + \rho')}\,U^2\,,$$

$$\gamma_e^{U} \approx \gamma\,(1 - 2\delta)\,\delta^2\,.$$

Thus, the effective surface tension according to Frenkel–Tonks γ_e^{σ} decreases as the squared flow velocity of the upper liquid with respect to the lower one, which is in good agreement with the results obtained in [9]. The physical meaning of γ_e^{σ} and γ_e^{U} is as follows: γ_e^{σ} is such a tension at which the Frenkel–Tonks instability arises at the initial value of the surface charge density σ. The same takes place for γ_e^{U}. Unlike γ_e^{σ}, γ_e^{U} was not introduced in the early works.

The relationships between γ and γ_e^{σ} as well as between γ and γ_e^{U} given in (17) make it possible to reduce the combined instability to the Frenkel–Tonks or Kelvin–Helmholtz instability.

NOTATION

A, B, C, D, constants, m/sec; g, acceleration of gravity, m/sec^2; F, constant, m^2/sec; i, imaginary unit; k, wave vector magnitude, m^{-1}; l, function of the wave vector magnitude, m^{-1}; p, pressure, Pa; t, time, sec; U, horizontal velocity of the upper liquid in infinity, m/sec; v', vector velocity of the upper liquid, m/sec; v_n and v'_n, normal component to the velocity interface between the lower and upper liquids, m/sec; v_x, v_z and v'_x, v'_z, horizontal and vertical velocity components of the lower and upper liquids, m/sec; W, function of the wave vector magnitude; x, horizontal axis; z, vertical axis; α, relative density coefficient; β, attenuation coefficient, sec^{-1}; $\tilde{\beta}$, dimensionless attenuation coefficient; γ, specific surface energy, J/m^2; γ_e^{U} and γ_e^{σ}, effective surface tension coefficients according to Kelvin–Helmholtz and Frenkel–Tonks; η, dynamic viscosity, Pa/sec; θ, dimensionless velocity; ξ, vertical coordinate of the interface, m; ρ and ρ', densities of the lower and upper liquids, kg/m^3; σ, charge density on the interface, C/m^2; σ_{xz}, component of the stress tensor, Pa; ν, kinematic viscosity, m^2/sec; Φ', velocity potential, m^2/sec; ω, complex frequency, sec^{-1}; Ω, function of the wave vector magnitude; ω', cyclic frequency, sec^{-1}; $\tilde{\omega}$, dimensionless cyclic velocity. Subscripts: e, effective.

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